

Improved Canonical Dual Algorithms for the Maxcut Problem

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Abstract By introducing a quadratic perturbation to the canonical dual of the maxcut problem, we transform the integer programming problem into a concave maximization problem over a convex positive domain under some circumstances, which can be solved easily by the well-developed optimization methods. Considering that there may exist no critical points in the dual feasible domain, a reduction technique is used gradually to guarantee the feasibility of the reduced solution, and a compensation technique is utilized to strengthen the robustness of the solution. The similar strategy is also applied to the maxcut problem with linear perturbation and its hybrid with quadratic perturbation. Experimental results demonstrate the effectiveness of the proposed algorithms when compared with other approaches.

Keywords Integer programming · canonical duality theory · reduction · compensation

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1 Introduction

As one of the 21 NP-hard combinatorial optimization problems [1], the maxcut problem has drawn a great deal of attention for several decades, due to its practical applications in circuit layout design, statistical physics, classification, and network analysis (for much more, the reader can refer to [2,3]).

There are special classes of graphs, for instance, the planar graphs and graphs with no K_5 minor, for which the maxcut problem can be solvable; however, it is impossible to find an algorithm in polynomial time due to its NP-hardness in general. Methods to solve the maxcut problem can be classified into two categories: direct and indirect. The direct method starts from a feasible solution, and then it ameliorates the solution iteratively with various strategies. Randomized heuristics which combine a greedy randomized adaptive search procedure, a variable neighborhood search, and a path-relinking intensification heuristic in [4], advanced scatter search in [5], a variant of spectral partitioning in [6], and a new tabu search algorithm in [7] belong to this kind. On the other hand, the indirect method focuses on the relaxation of the primal or the Lagrangian dual of the maxcut models, and then it uses some procedures to make the relaxation solution feasible to original problem, in which, relaxation means that relaxing some of the constraints and extending the objective function to larger space. Based on the semidefinite programming (SDP) relaxation, Goemans and Williamson [8] proposed the well-known randomized approximation algorithm which can achieve a performance guarantee of 0.878. Based on the rank-two relaxation, a continuous optimization heuristic was constructed for approximating the maxcut problem [9]. Based on the Lagrangian dual relaxation, smoothing heuristics were presented in [10]. In [11], a polyhedral cut and price approach was investigated to solve the maxcut problem, in which, at the pricing phase, an interior point cutting plane algorithm was used to solve the Lagrangian dual of the SDP relaxation; at the cutting phase, cutting planes based on the polyhedral theory were added to the primal problem in order to improve the the SDP relaxation. In [12], a feasible direction method was designed to solve the continuous nonlinear programming problem by the relaxed maxcut model. In a branch and bound setting, a dynamic bundle method was used to solve the Lagrangian dual of the semidefinite maxcut relaxation[13].

The canonical dual is a generalized Lagrangian dual, which originated in the late 1980s by Gao and Strang, and has developed significantly in recent years, which provides a new and potentially useful methodology for solving integer programming problems [14,15]. As shown in [16,17], through special canonical dual transformation, the integer programming problems can be converted to a concave maximization problem over a convex positive domain without duality gap, which can be solved easily by well-developed optimization algorithms under some circumstances. In [18], a canonical dual approach was proposed for the maxcut problem by adding a linear perturbation term to the primal objective function; however, the strategy is not so robust to guarantee the solution to be located in the primal feasible domain. In this paper, we

introduce a quadratic perturbation to the canonical dual of the maxcut problem. Furthermore, for some cases when there exist no critical points in the dual feasible space, a reduction technique is used gradually to guarantee the feasibility of the reduced solution and a compensation technique is utilized to strengthen the robustness of the solution. We also apply the strategy to the maxcut problem with linear perturbation and its hybrid with quadratic perturbation. Finally, experimental results are given to testify the effectiveness of the proposed algorithms.

2 The Maxcut Problem

Let $G = (V, E)$ be an undirected graph with edge weight w_{ij} on $n + 1 = |V|$ vertices and $m = |E|$ edges, for each edge $(i, j) \in E$, the maxcut problem is to find a subset S of the vertex set V such that the total weight of the edges between S and its complementary subset $\bar{S} = V \setminus S$ is as large as possible.

2.1 LP based maxcut model

Considering a variable y_{ij} for each edge $(i, j) \in E$, and assuming y_{ij} to be 1 if (i, j) is in the cut, and 0 otherwise, the maxcut problem can be modeled as the following linear programming (LP) optimization problem:

$$\begin{aligned} \max \quad & W(\mathbf{y}) = \sum_{i=1}^{n+1} \sum_{i < j, (i,j) \in E} w_{ij} y_{ij} \\ \text{s.t.} \quad & \mathbf{y} \text{ is the incidence vector of a cut.} \end{aligned} \quad (1)$$

Here the incidence vector $\mathbf{y} = \{y_{ij}\} \in \mathbb{R}^m$, where the m is the number of edges in the graph.

Let $\text{CUT}(G)$ denote the convex hull of the incidence vectors of cuts in G . Since maximizing a linear function over a set of points equals to maximizing it over the convex hull of this set of points, we can rewrite (1) to the following form:

$$\begin{aligned} \max \quad & W(\mathbf{y}) = \mathbf{c}^T \mathbf{y} \\ \text{s.t.} \quad & \mathbf{y} \in \text{CUT}(G). \end{aligned} \quad (2)$$

where $\mathbf{c} = \{w_{ij}\} \in \mathbb{R}^m$.

2.2 SDP based maxcut model

For a bipartition (S, \bar{S}) , with $y_i = 1$ if $i \in S$, and $y_i = -1$ otherwise, the maxcut problem can be formulated as the following semidefinite programming

optimization problem:

$$\begin{aligned} \max \quad & W(\mathbf{y}) = \frac{1}{4} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} w_{ij} (1 - y_i y_j) \\ \text{s.t.} \quad & \mathbf{y} \in \{-1, 1\}^{n+1}. \end{aligned} \quad (3)$$

Without loss of generality, if we fix the value of the last variable at 1, then the problem (3) is equivalent to the following form (primal problem):

$$(\mathcal{P}) : \min \left\{ P(\mathbf{x}) = \frac{1}{2} \langle \mathbf{x}, Q\mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{c} \rangle : \mathbf{x} \in \{-1, 1\}^n \right\}, \quad (4)$$

where, $\langle \mathbf{u}, \bar{\mathbf{u}} \rangle = \mathbf{u}^T \bar{\mathbf{u}}$ denotes the bilinear form, Q is a symmetric matrix with $Q_{ij} = w_{ij} (i, j = 1, 2, \dots, n)$, and $\mathbf{c} = -(w_{1(n+1)}, \dots, w_{n(n+1)})^T$. It is not difficult to find that a optimal solution \mathbf{x}^* to problem (4) corresponds to a optimal solution $(\mathbf{x}^*, 1)$ of original problem (3).

The canonical dual algorithms in this paper are based on the SDP model.

3 Primal problem with linear perturbation

In [18], a linear non-zero vector $\Delta \mathbf{c} = (\Delta c_1, \dots, \Delta c_n) \in \mathbb{R}^n$ is added to the primal problem and then we can get the linearly perturbed problem

$$(\mathcal{P}_l) : \min \left\{ P(\mathbf{x}) = \frac{1}{2} \langle \mathbf{x}, Q\mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{c} + \Delta \mathbf{c} \rangle : \mathbf{x} \in \{-1, 1\}^n \right\}. \quad (5)$$

It proved that, if $\sum_{i=1}^n |\Delta c_i| < 1$, the perturbed optimal solution is equivalent to the solution of primal problem, and if $\sum_{i=1}^n |\Delta c_i| = \alpha W(\tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}}$ is the optimal solution of the perturbed problem and $\alpha > 0$ is a constant, then $W(\tilde{\mathbf{x}}) \geq \frac{1}{1+\alpha} W^*$, here, W^* is the optimal value of the primal problem.

The problem of this linear perturbation is that, for one thing, it is not easy to find such an appropriate perturbation, and for another, if $\sum_{i=1}^n |\Delta c_i| > 1$, only approximate solution can be obtained.

4 Canonical dual problem with quadratic perturbation

We give the notations $\mathbf{s} \circ \mathbf{t} := [s_1 t_1, \dots, s_m t_m]$ and $\mathbf{s} \oslash \mathbf{t} := [s_1/t_1, \dots, s_m/t_m]$ to denote the Hadamard product and quotient for any two vector $\mathbf{s}, \mathbf{t} \in \mathbb{R}^m$, $Diag(\boldsymbol{\nu})$ to represent a diagonal matrix with components of the vector $\boldsymbol{\nu}$ as its elements, and \mathbf{e} to stand for a vector of all ones.

By the fact $\mathbf{x} \circ \mathbf{x} = \mathbf{e}$ and $[\mathbf{x} \circ \mathbf{x}]^T \boldsymbol{\alpha} = \mathbf{x}^T Diag(\boldsymbol{\alpha}) \mathbf{x} = \mathbf{e}^T \boldsymbol{\alpha}$, then the problem (\mathcal{P}) is identical to the following α -perturbed problem:

$$(\mathcal{P}_\alpha) : \min \left\{ P_\alpha(\mathbf{x}) = \frac{1}{2} \langle \mathbf{x}, (Q + Diag(\boldsymbol{\alpha})) \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{c} \rangle - d_\alpha : \mathbf{x} \in \{-1, 1\}^n \right\}, \quad (6)$$

where $d_\alpha = \frac{1}{2}\langle \mathbf{e}, \boldsymbol{\alpha} \rangle$, $\boldsymbol{\alpha} \in \mathbb{R}^n$ is a parametrical vector.

Introducing a quadratic geometrical operator

$$\boldsymbol{\epsilon} = \Lambda(\mathbf{x}) = \frac{1}{2}(\mathbf{x} \circ \mathbf{x} - \mathbf{e}) \quad (7)$$

and following the standard procedures of canonical dual methodology, the α -perturbed canonical dual problem is obtained as follows

$$(\mathcal{P}_\alpha^d) : \max \left\{ P_\alpha^d(\boldsymbol{\sigma}) = -\frac{1}{2}\langle G_\alpha^{-1}(\boldsymbol{\sigma})\mathbf{c}, \mathbf{c} \rangle - \frac{1}{2}\langle \mathbf{e}, \boldsymbol{\sigma} \rangle - d_\alpha \mid \boldsymbol{\sigma} \in \mathcal{S}_\alpha^+ \right\}, \quad (8)$$

where, $G_\alpha(\boldsymbol{\sigma}) = Q + \text{Diag}(\boldsymbol{\alpha} + \boldsymbol{\sigma})$, and

$$\mathcal{S}_\alpha^+ = \{\boldsymbol{\sigma} \in \mathbb{R}^n \mid G_\alpha(\boldsymbol{\sigma}) \succ 0\}. \quad (9)$$

Further more, a quadratic penalty term is added to the α -perturbed canonical dual problem, and then we get the β -perturbed canonical dual problem

$$(\mathcal{P}_{\alpha\beta}^d) : \max \left\{ P_{\alpha\beta}^d(\boldsymbol{\sigma}) = -\frac{1}{2}\langle G_\alpha^{-1}(\boldsymbol{\sigma})\mathbf{c}, \mathbf{c} \rangle - \frac{1}{2} \sum_{i=1}^n \left(\frac{\sigma_i^2}{\beta_i} + \sigma_i \right) - d_\alpha \mid \boldsymbol{\sigma} \in \mathcal{S}_\alpha^+ \right\}. \quad (10)$$

Theorem 1 Suppose that for a given perturbation vector $\boldsymbol{\alpha}$ such that $Q + \text{Diag}(\boldsymbol{\alpha}) \prec 0$, and $\beta_i \gg 0$ ($i = 1, 2, \dots, n$) is big enough, if $\bar{\boldsymbol{\sigma}} \in \mathcal{S}_\alpha^+$ is a critical point of the β -perturbed canonical dual problem, then

$$\bar{\mathbf{x}} = G_\alpha^{-1}(\bar{\boldsymbol{\sigma}})\mathbf{c}, \mathbf{x}^* = \text{round}(\bar{\mathbf{x}}), \quad (11)$$

\mathbf{x}^* is a global solution to the (\mathcal{P}) .

Proof First, we prove that \mathbf{x}^* defined by (11) is feasible of (\mathcal{P}) .

Considering that

$$\begin{aligned} \frac{\partial P_{\alpha\beta}^d(\boldsymbol{\sigma})}{\partial \bar{\sigma}_i} &= -\frac{1}{2} \frac{\partial [\mathbf{c}^T G_\alpha^{-1}(\boldsymbol{\sigma})\mathbf{c}]}{\partial \bar{\sigma}_i} - \frac{\bar{\sigma}_i}{\beta_i} - \frac{1}{2} \\ &= \frac{1}{2} \mathbf{c}^T G_\alpha^{-1}(\bar{\boldsymbol{\sigma}}) \frac{\partial G_\alpha(\boldsymbol{\sigma})}{\partial \bar{\sigma}_i} G_\alpha^{-1}(\bar{\boldsymbol{\sigma}}) \mathbf{c} - \frac{\bar{\sigma}_i}{\beta_i} - \frac{1}{2} \\ &= \frac{1}{2} (\bar{\mathbf{x}} \circ \bar{\mathbf{x}})_i - \frac{\bar{\sigma}_i}{\beta_i} - \frac{1}{2} \\ &= \frac{1}{2} \bar{x}_i^2 - \frac{\bar{\sigma}_i}{\beta_i} - \frac{1}{2}, \end{aligned}$$

if $\bar{\boldsymbol{\sigma}}$ is a critical point of $(\mathcal{P}_{\alpha\beta}^d)$, then $\frac{1}{2} \bar{x}_i^2 - \frac{\bar{\sigma}_i}{\beta_i} - \frac{1}{2} = 0$, namely, $\bar{x}_i^2 = \frac{2\bar{\sigma}_i}{\beta_i} + 1$.

If $\beta_i \gg 0$ is big enough, it is easy to find that $x_i^* = \text{round}(\bar{x}_i) = \pm 1$, which is a feasible solution to (\mathcal{P}) .

Then, we prove that \mathbf{x}^* is a global solution.

By the fact that $Q + \text{Diag}(\boldsymbol{\alpha}) \prec 0$, we can conclude that $\bar{\boldsymbol{\sigma}} > 0$ due to $G_\alpha(\bar{\boldsymbol{\sigma}}) \succ 0$. Therefore, the β -perturbed canonical dual problem $(\mathcal{P}_{\alpha\beta}^d)$ is

strictly concave on \mathcal{S}_α^+ , and if the critical point $\bar{\sigma} \in \mathcal{S}_\alpha^+$, it must be a unique maximizer of problem $(\mathcal{P}_{\alpha\beta}^d)$. Consequently, the corresponding \mathbf{x}^* defined by (11) must be unique. On the other hand, we have

$$\begin{aligned} P_\alpha(\bar{\mathbf{x}}) &= \frac{1}{2} \langle \bar{\mathbf{x}}, (Q + \text{Diag}(\alpha)) \bar{\mathbf{x}} \rangle - \langle \bar{\mathbf{x}}, \mathbf{c} \rangle - d_\alpha \\ &= \frac{1}{2} \langle \bar{\mathbf{x}}, G_\alpha(\bar{\sigma}) \bar{\mathbf{x}} \rangle - \langle \bar{\mathbf{x}}, \mathbf{c} \rangle - \frac{1}{2} \langle \bar{\sigma}, \mathbf{e} \rangle - d_\alpha \\ &= \min_{\mathbf{x} \in \mathbb{R}^n} \frac{1}{2} \langle \mathbf{x}, G_\alpha(\bar{\sigma}) \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{c} \rangle - \frac{1}{2} \langle \bar{\sigma}, \mathbf{e} \rangle - d_\alpha \\ &\leq \min_{\mathbf{x} \in \{-1, 1\}^n} \frac{1}{2} \langle \mathbf{x}, (Q + \text{Diag}(\alpha)) \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{c} \rangle - d_\alpha, \end{aligned}$$

that is to say, $\mathbf{x}^* = \text{round}(\bar{\mathbf{x}})$ is the global solution of (\mathcal{P}_α) and so the global solution of (\mathcal{P}) .

5 Canonical dual algorithms

Now, we present a gradient-based iterative method for the β -perturbed canonical dual problem.

It is not difficult to find that the search direction (negative gradient) of the negative canonical dual function $-P_{\alpha\beta}^d(\sigma)$ is

$$d = \frac{1}{2} \mathbf{x} \circ \mathbf{x} - \sigma \oslash \beta - \frac{1}{2} \mathbf{e}, \quad (12)$$

and step size of each component of search direction is obtained by a golden section search technique from $[0, \alpha_{\max}]$, where α_{\max} is determined by making the corresponding diagonal component of $G_\alpha(\sigma)$ positive.

We use three types of termination criteria to stop the procedure. The details of the algorithm is give in the following

Algorithm 1

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1: Initialization, set  $\sigma_k = \sigma^0$ ,  $\alpha$ ,  $\beta$ ,  $\epsilon$  and  $k = 0$ 
2: while (1) do
3:    $\mathbf{x}_k \leftarrow (Q + \text{Diag}(\alpha + \sigma_k))^{-1} \mathbf{c}$ ,  $\mathbf{d}_k \leftarrow \frac{1}{2} \mathbf{x}_k \circ \mathbf{x}_k - \sigma_k \oslash \beta - \frac{1}{2} \mathbf{e}$ 
4:    $\sigma_{k+1} = \sigma_k + a_k \times \mathbf{d}_k$ 
5:    $\mathbf{x}_{k+1} \leftarrow (Q + \text{Diag}(\alpha + \sigma_{k+1}))^{-1} \mathbf{c}$ 
6:   if  $\|\mathbf{d}_k\| \leq \epsilon$  or  $\|\frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\mathbf{x}_k}\| \leq \epsilon$  or  $\|\frac{\sigma_{k+1} - \sigma_k}{\sigma_k}\| \leq \epsilon$  then
7:      $\mathbf{x}^* \leftarrow \text{round}(\mathbf{x}_{k+1})$ , break
8:   end if
9:    $k \leftarrow k + 1$ 
10: end while
```

where, ϵ is the numerical precision, and the initial point σ^0 should make $G_\alpha(\sigma^0)$ positive.

Remark 1 If the final critical point σ^* of $(\mathcal{P}_{\alpha\beta}^d)$ is in \mathcal{S}_α^+ , according to Theorem

1, we can conclude that the corresponding \mathbf{x}^* is the global solution to (\mathcal{P}) .

However, the critical point σ^* is not always in \mathcal{S}_α^+ , or in \mathcal{S}_α^+ , we can not find a critical point σ^* . In this case, we can find that some components of \mathbf{x}^* are feasible, and other components are infeasible. Thus, there must exist a transformation $N \in \mathbb{R}^{n \times m}$, such that

$$\mathbf{x}^* = \mathbf{x}_p + N\mathbf{x}_h, \quad (13)$$

where, $\mathbf{x}_p \in \mathbb{R}^n$ is the particular solution with infeasible components zeros, and $\mathbf{x}_h \in \mathbb{R}^m$ is called the reduced solution.

Considering that

$$\begin{aligned} P(\mathbf{x}^*) &= P(\mathbf{x}_p + N\mathbf{x}_h) \\ &= \frac{1}{2}(\mathbf{x}_p + N\mathbf{x}_h)^T Q(\mathbf{x}_p + N\mathbf{x}_h) - c^T(\mathbf{x}_p + N\mathbf{x}_h) \\ &= \frac{1}{2}\mathbf{x}_h^T (N^T Q N) \mathbf{x}_h - \mathbf{x}_h^T (N^T \mathbf{c} - N^T Q \mathbf{x}_p) + \frac{1}{2}\mathbf{x}_p^T Q \mathbf{x}_p - c^T \mathbf{x}_p, \end{aligned} \quad (14)$$

as a result, we have to solve the reduced problem in further

$$P(\mathbf{x}_h) = \frac{1}{2}\langle \mathbf{x}_h, Q_h \mathbf{x}_h \rangle - \langle \mathbf{x}_h, \mathbf{c}_h \rangle, \quad (15)$$

where, $Q_h = N^T Q N$, $\mathbf{c}_h = N^T \mathbf{c} - N^T Q \mathbf{x}_p$.

The reduction technique above can be regarded as “greedy criterion”, which belongs to local search to some extent. To compensate for the lost information in the reduction process, we use a simple compensation technique, that is, replacing each binary component in \mathbf{x}^* with its counterpart, to strength the robustness of the solution. The details of the improved canonical dual algorithm are given below:

Algorithm CDA1

- 1: Using Algorithm 1 as a subroutine to get \mathbf{x}^*
 - 2: **if** the corresponding $\sigma^* \notin \mathcal{S}_\alpha^+$ **then**
 - 3: obtaining current \mathbf{x}_h by the reduction technique
 - 4: **while** $\text{length}(\mathbf{x}_h) \geq 1$ **do**
 - 5: solving the current reduced problem $P(\mathbf{x}_h)$ by Algorithm 1 again
 - 6: **if** the corresponding $\sigma_h^* \in \mathcal{S}_{\alpha(h)}^+$ **then**
 - 7: $\mathbf{x}^* = \mathbf{x}_p + N\mathbf{x}_h$, **break**
 - 8: **else**
 - 9: obtaining current \mathbf{x}_h by the reduction technique again
 - 10: **end if**
 - 11: **end while**
 - 12: **end if**
 - 13: Using the compensation technique to strength the robustness of the solution
-

We also apply the similar reduction technique and compensation technique to the maxcut problem with linear perturbation in [18] and its hybrid with quadratic perturbation in this paper, so we get the corresponding algorithms named CDA2 and CDA3.

6 Experimental results

To testify the effectiveness of the proposed algorithms, firstly, we give a random example to show that in some case, the Algorithm 1 can achieve a global solution without additional strategies. Then we compared our algorithms with other approaches for some published TSPLIB (see in [20]) instances results. Finally, some large instances in TSPLIB are tested to show the capacity of our algorithms. We run the proposed procedures in MATLAB R2010b on Intel(R) Core(TM) i3-2310M CPU @2.10GHz under Window 7 environment.

Example 1 Consider the following 10-dimensional problem($n=9$) with randomly selected matrix Q and \mathbf{c} , $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$Q = \begin{pmatrix} 0 & 6 & 4 & 8 & 4 & 5 & 5 & 6 & 8 \\ 6 & 0 & 3 & 9 & 4 & 5 & 5 & 8 & 7 \\ 4 & 3 & 0 & 6 & 2 & 4 & 7 & 5 & 4 \\ 8 & 9 & 6 & 0 & 7 & 4 & 7 & 6 & 6 \\ 4 & 4 & 2 & 7 & 0 & 7 & 5 & 4 & 6 \\ 5 & 5 & 4 & 4 & 7 & 0 & 0 & 2 & 7 \\ 5 & 5 & 7 & 7 & 5 & 0 & 0 & 4 & 5 \\ 6 & 8 & 5 & 6 & 4 & 2 & 4 & 0 & 2 \\ 8 & 7 & 4 & 6 & 6 & 7 & 5 & 2 & 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 5 \\ 2 \\ 5 \\ 4 \\ 5 \\ 7 \end{pmatrix}, \boldsymbol{\alpha} = \begin{pmatrix} -17.3208 \\ -2.8050 \\ -36.5410 \\ -1.1174 \\ -38.3706 \\ -77.0470 \\ -20.1651 \\ -31.3002 \\ -34.9461 \end{pmatrix}, \boldsymbol{\beta} = \begin{pmatrix} 605.7162 \\ 601.1675 \\ 330.2360 \\ 277.4284 \\ 674.9582 \\ 540.0750 \\ 537.7345 \\ 690.3018 \\ 371.8627 \end{pmatrix}$$

we can get the dual critical solution

$$\boldsymbol{\sigma}^* = (34.0286, 19.9327, 50.1747, 12.6699, 55.9428, 88.7105, 30.2908, 45.0242, 45.6742)^T,$$

and the corresponding

$$\mathbf{x}^* = \text{round}(G^{-1}(\bar{\boldsymbol{\sigma}})\mathbf{c}) = (1, 1, 1, -1, 1, -1, -1, -1, -1).$$

We can check that $\boldsymbol{\sigma}^*$ is in the dual feasible domain \mathcal{S}_α^+ , that is to say, the corresponding \mathbf{x}^* is the global solution according to Theorem 1.

Example 2 We compare our algorithms with the published results found in [8](*GW*) and [9, 19](*circut*). From Table 1, we find that all of the canonical dual algorithms can achieve the same or better solutions than that of (*circut*) and (*GW*), with comparable shorter time.

Table 1 Comparison with other algorithms for medium-size TSPLIB instances

Instances	<i>circut</i>	time	<i>GW</i>	time	<i>CDA1</i>	time	<i>CDA2</i>	time	<i>CDA3</i>	time
burma14	283	0.046	-	-	283	0.216	283	0.084	283	0.095
gr17	24986	0.047	-	-	24986	0.087	24986	0.272	24986	0.189
bays29	53990	1.109	-	-	53990	0.349	53990	0.159	53990	0.152
dantzig42	42638	1.75	42638	43.35	42638	0.540	42638	0.204	42638	0.203
gr48	320277	3.672	320277	26.17	320277	0.346	320277	0.287	320277	0.294
hk48	771712	2.516	771712	66.52	771712	1.368	771712	0.426	771712	0.510
gr96	105328	14.250	105295	531.50	105328	0.550	105328	0.975	105328	0.693
kroA100	5897368	2.359	5897392	420.83	5897392	1.444	5897392	1.053	5897392	0.785
kroB100	5763020	2.531	5763047	917.47	5763047	1.440	5763047	1.021	5763047	0.890
kroC100	5890745	2.500	5890760	398.78	5890760	1.258	5890760	0.986	5890760	0.898
kroD100	5463250	2.547	5463250	469.48	5463250	1.571	5463250	1.115	5463250	0.843
kroE100	5986587	2.500	5986591	375.68	5986591	1.476	5986591	1.476	5986591	0.888
gr120	-	-	2156667	754.87	2156667	2.752	2156667	1.978	2156667	1.358

Example 3 Other instances from TSPLIB are used to test the capacity of the proposed algorithms, and we test the size of the instances up to 500 with no more than 2 minutes. In the meanwhile, we use a toolbox called YALMIP [21] to solve the same problems for comparison. The results also show the different performance of the three canonical dual algorithms ($CDA3 > CDA1 > CDA2$), which indicate that the hybrid with linear and quadratic perturbation is a better choice, especially for gil262.

Table 2 Comparison with YALMIP for large-size TSPLIB instances

Instances	<i>YALMIP</i>	time	<i>CDA1</i>	time	<i>CDA2</i>	time	<i>CDA3</i>	time
ch130	1888108	3.448	1888108	3.036	1888108	2.608	1888108	1.461
ch150	2525606	4.007	2525626	3.767	2525626	3.400	2525626	1.671
d198	12938532	6.910	12938532	9.256	12938532	16.725	12938532	3.790
gr202	195433	6.980	197098	7.600	197098	8.196	197098	4.866
gr229	1203249	9.463	1205180	9.513	1205180	7.697	1205180	8.576
gil262	2131227	13.360	2149623	10.479	2147693	10.352	2152173*	6.892
pr299	80047271	20.156	80324674	34.453	80324674	19.710	80324674	14.165
lin318	59504077	24.221	59547803	17.888	59547803	33.739	59547803	13.158
fl417	76742924	61.888	76776296	28.738	76776296	40.181	76776296	22.590
pr439	277700818	73.775	278552199	46.210	278552199	55.039	278552199	37.913
d493	75651106	99.701	75740087	53.323	75740087	66.291	75740087	45.986
att532	281648797	127.485	287417240	57.681	287417240	109.979	287417240	52.192

7 Conclusion

In this paper, we have studied the maxcut problem with a quadratic perturbation term added to the canonical dual problem. A gradient-based algorithm is designed to solve the quadratic perturbed canonical dual problem. For some cases with no critical points in the dual feasible domain, a reduction technique and a compensation technique are introduced to improve the quality of the solution. The similar techniques are also applied to maxcut model with linear perturbation and its hybrid with quadratic perturbation. Numerical results show that the proposed algorithms are effective in terms of both time complexity and the solution quality, in the same time, it indicates that the hybrid perturbation is a more promising approach.

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